DISCRETE MATHEMATICS

ELSEVIER Discrete Mathematics 179 (1998) 133–143

The centroidal branches of a separable graph are edge reconstructible

Robert Molina*

Department of Mathematics and Computer Science, Alma College, Alma, MI 48801, USA

Received 25 July 1994; received in revised form 14 May 1996; accepted 11 November 1996

Abstract

If T is a tree, then the weight of a vertex v in T is the number of vertices in a largest component of $T-v$. The centroid of a tree is the set of vertices of minimum weight. We show that if G is a separable graph then there is a unique block or cutvertex that contains the centroids of all spanning trees of G. We define this block or cutvertex to be the centroid of G. We show that the centroid and rooted branches of the centroid are edge reconstructible, that is, determined up to isomorphism by the set of edge-deleted subgraphs.

1. Introduction

All graphs in this paper are finite, simple and undirected. Let G be a graph. We will denote the number of vertices in G by $|G|$. A vertex v is a *cutvertex* if $G-v$ has more components than G, and an edge e is a *bridge* if $G-e$ has more components than G. An edge which is incident with a degree 1 vertex is an *endline*. A connected graph with at least one cutvertex is *separable* and a connected nontrivial graph without cutvertices is *nonseparable*. A *block* of a graph is a maximal nonseperable subgraph. In a tree, for example, all blocks are isomorphic to K_2. Blocks that contain more than one edge are *cyclic blocks*. If G is a separable graph with a block A then a *branch at* A is a maximal connected subgraph of G whose intersection with A is a single vertex. If v is a cutvertex of G then a *branch at* v is a maximal connected subgraph of G in which v is not a cutvertex. If B is a branch at either a cutvertex or block then we define the *weight* of B, $w(B)$, by $w(B)=|B|-1$. Note that the number of vertices in G is equal to the number of vertices in a given block (cutvertex) plus the sum of the weights of the branches at that block (cutvertex). The *block-cutvertex tree* of a connected graph G, $BC(G)$, is the tree that has the blocks and cutvertices of G as its

* E-mail: molina@alma.edu.

PII S0012-365X(97)00032-0

vertex set, with adjacencies occurring between vertices if and only if one vertex is a block, the other is a cutvertex, and the block contains the cutvertex.

The *edge deck* of G, denoted $ED(G)$, is the set of edge deleted subgraphs of G, and we refer to the elements of $ED(G)$ as *edge cards*. A graph H is an *edge reconstruction* of G if $ED(G)=\{G-e_i\}_{i=1}^n$, $ED(H)=\{H-e_i'\}_{i=1}^n$, and $G-e_i \approx H-e_i'$ for all i. G is *edge reconstructible* if every edge reconstruction of G is isomorphic to G, and a family of graphs is edge reconstructible if every graph in the family is edge reconstructible. Examples of edge reconstructible families of graphs include regular graphs, disconnected graphs and trees [3,2]. The edge reconstruction conjecture states that all graphs with at least four edges are edge reconstructible. A graph invariant is edge reconstructible if it is determined by the edge deck. A fundamental result in edge reconstruction is Kelly's Lemma [3,2] which states that if G and H are graphs and H has fewer edges than G, then the number of subgraphs of G that are isomorphic to H is edge reconstructible. Vertex reconstructibility of a graph is defined in a similar manner, with vertices deleted instead of edges, and the vertex reconstruction conjecture states that all graphs with at least 3 vertices are vertex reconstructible. Greenwell showed that vertex reconstructibility of a graph implies edge reconstructibility [3,2]. For a survey of early results on the reconstruction problem, see [3]. More recent results are summarized in [2].

The edge reconstruction of separable graphs is an open problem. Most of the results on the reconstruction of separable graphs have been vertex reconstruction results, and surprisingly little attention has been given to edge reconstruction of such graphs. An early result of Bondy [1] is that separable graphs without endlines are vertex reconstructible (which by Greenwell's result implies edge reconstructibility). Yang has shown that if all nonseparable graphs are vertex reconstructible, then so are the separable ones [6]. In Section 2 we define centroid and centroidal branches of a separable graph. The main result of this paper is that the centroid and centroidal branches of a separable graph are edge reconstructible. Myrvold [5] was the first to use the centroid for reconstruction purposes, and uses the centroid to compute the ally-reconstruction number of a tree. In [4], Greenwell and Hemminger prove a result that is similar to ours and use the vertex deck to reconstruct the branches of the pruned center of a separable graph. (The centroid and pruned center of a separable graph are both blocks, but they need not be the same.) The result in [4] does not apply to arbitrary separable graphs, but to graphs G with a degree one vertex v such that $G-v$ has at least two branches at the pruned center with endlines. Our result on edge reconstruction of centroidal branches applies to arbitrary separable graphs.

2. The centroid of a separable graph

If v is a vertex of a tree T then the *weight* of v is defined to be the number of vertices in a largest component of $T-v$. The *centroid* of a tree is its set of vertices of minimal weight, and we refer to vertices in the centroid as *centroidal vertices*.

We make some observations about the centroid of a tree.

Observation 1. *The centroid of a tree contains either a single vertex or a pair of adjacent vertices. We say that T is unicentroidal or bicentroidal depending on the size of the centroid.*

Observation 2. *A tree T is unicentroidal with centroidal vertex c if and only if all components of $T-c$ have order at most $|T-c|/2$.*

Observation 3. *A tree T is bicentroidal if and only if there exists an edge e such that $T-e$ has two components of order $|T|/2$.*

We wish to extend the definition of centroid so that it will apply to separable graphs. The next proposition gives the motivation for this definition.

Proposition 1. *If G is a separable graph, then there is a block of G that contains the centroids of all spanning trees of G.*

Proof. Let T_1 and T_2 be spanning trees of G where c_1 is a centroidal vertex of T_1 and c_2 is a centroidal vertex of T_2. Suppose that c_1 and c_2 do not lie in a common block. Then there is a cutvertex v in G that lies between c_1 and c_2. Let B be the branch of G at v that contains c_2. Let B_1 be the part of T_1 contained in B and let B_2 be the part of T_2 contained in B. Let T be the tree obtained from T_1 by replacing the branch B_1 with B_2. Then T is a spanning tree of G. Since the weights of the branches of c_1 in T are the same as the weights of the branches of c_1 in T_1, c_1 is a centroidal vertex of T. Similarly, the weights of the branches of c_2 in T are the same as the weights of the branches of c_2 in T_2, and thus c_2 is also a centroidal vertex of T. But this is impossible since c_1 and c_2 are not adjacent in T. Thus, c_1 and c_2 must lie in a common block. Since T_1 and T_2 were arbitrary spanning trees of G it follows that there is a block B of G that contains the centroidal vertices of every spanning tree of G. $\square$

By Proposition 1 there are two possibilities for a separable graph G. Either there is a unique block B that contains the centroid of every spanning tree of G, or there is a unique cutvertex c that is the centroid of every spanning tree of G. We define the *centroid* of a separable graph G to be the unique block or cutvertex of G that contains the centroidal vertices of all spanning trees of G. If the centroid is a block then we say that G is *block centroidal* and if the centroid is a cut vertex then we say that G is *vertex centroidal*. If G is block centroidal and the centroid has exactly one edge then we say that G is *edge centroidal*. The following observations follow directly from related facts about centroids of trees and the natural correspondence between the branches of a separable graph and the branches in a spanning tree of that graph.

Observation 4. *A separable graph G is vertex centroidal with centroid c if and only if the weight of each branch at c is at most $(|G|-1)/2$.*

Observation 5. *A separable graph G is block centroidal with centroid C if and only if the weight of each branch at C is less than $(|G|-1)/2$.*

We will refer to branches of the centroid as *centroidal branches*. When trying to reconstruct the centroidal branches of a separable graph G we will use the centroid of G as a point of reference. The following observations will be used frequently when trying to identify the centroid of G on an edge card $G-e$.

Observation 6. *If G is a vertex centroidal separable graph and e is an edge of G that is not a bridge, then the centroid of G is also the centroid of $G-e$.*

Observation 7. *Let G be a block centroidal separable graph that contains an edge e that is not a bridge. Then the centroid of G is also the centroid of $G-e$ unless e is in the centroid of G, in which case the centroid of $G-e$ is properly contained in the centroid of G.*

Observation 8. *Let G be a vertex centroidal separable graph with centroid c and let e be an endline in G. Then c is the centroid of $G-e$ unless there is a centroidal branch B of weight $(|G|-1)/2$ that does not contain e, in which case the centroid of $G-e$ is the block of B containing the vertex c.*

Observation 9. *Let G be a block centroidal separable graph with centroid C, and let e be an endline in G. Then C is the centroid of $G-e$ unless there is a centroidal branch B of weight $(|G|/2)-1$ that does not contain e, in which case the centroid of $G-e$ is that cutvertex common to both B and C.*

If e is an endline of G then we will refer to $G-e$ as an *end card*. Observations 8 and 9 essentially say that deleting an endline leaves the centroid fixed unless there is a centroidal branch of weight approximately $|G|/2$, in which case the centroid may shift slightly in the direction of that branch. If e is an edge on a cycle that is not in the centroid of a separable graph G then we will refer to $G-e$ as a *cycle card*. If we know that G has at least one cycle outside of the centroid then by Observations 6 and 7 the cycle cards can be recognized as those connected cards whose centroids contain a maximum number of edges. The reconstruction of the centroidal branches of G will depend almost exclusively on the use of cycle cards and end cards.

3. Reconstruction of centroidal branches

We wish to show that the centroid and centroidal branches of a separable graph are reconstructible from $ED(G)$. As we noted in the introduction, trees and separable graphs without endlines are edge reconstructible. Throughout the rest of this section, let G be a separable graph with at least one endline and at least one cycle.

Proposition 2. *The centroid of G is edge reconstructible.*

Proof. The number of cyclic blocks in G is just the number of cyclic blocks appearing on any end card. If G has at least two cyclic blocks then $ED(G)$ has cycle cards and the centroid of G is just the centroid on any cycle card. So suppose that G has only one cyclic block A. If G has just one endline then clearly A is the centroid of G. We next consider the case where there are exactly two edges of G that are not in A. Then $|G|=|A|+2$ and A is the centroid unless $|A|=3$. There are only three such graphs with $|A|=3$, and it is easily verified that these are edge reconstructible. Thus, we may assume that there are three or more edges of G that are not in A. Then G has two or more branches at A if and only if there exists an end card with two or more branches at A. If there are two or more branches at A, then a branch of maximum weight will appear on some end card. If there is just one branch at A then the weight of that branch is $|G|-|A|$. Since the weight of a largest branch at A is reconstructible, we can use Observation 5 to determine whether A is the centroid of G. If A is not the centroid then G is edge centroidal if there is an edge card with two components of the same order, and vertex centroidal otherwise. Thus, the centroid of G is reconstructible. □

Because the centroid is reconstructible, the cycle cards in a graph are recognizable since they are the connected cards in the edge deck with the same centroid as G. If G has cycle cards, then the number of centroidal branches is reconstructible. For if G is block centroidal then any cycle card will have the same number of centroidal branches as G, and if G is vertex centroidal then any cycle card with a minimum number of centroidal branches will have the same number of centroidal branches as G.

If B is a centroidal branch of G and v is the vertex of B that is in the centroid of G, then we say that v is the *root* of B and refer to $B(v)$ as a *rooted centroidal branch* of G. In the following arguments when we say that the centroidal branches of a graph are reconstructible, we mean the rooted centroidal branches.

Proposition 3. *If G has at least two centroidal branches that contain cycles, then the centroidal branches of G are edge reconstructible.*

Proof. Assume that G has at least two centroidal branches that contain cycles. We first show that any reconstruction of G has two centroidal branches that contain cycles. This is the case if there exists a cycle card with the same number of centroidal branches as G, and two of these branches contain cycles. Otherwise, G has exactly two centroidal branches containing cycles, and each branch contains just one cycle.

We first assume the centroid of G is a cyclic block. Let e be an endline of G, and let $G'=G-e+e'$ be a reconstruction of G. If the centroid of G is not a cycle, then since it is the only such block on $G-e$, it must also be the centroid in G'. Suppose the centroid of G is a cycle. Since $G-e$ has three cycles, any reconstruction of G has two cycles outside the centroid. We claim that in any reconstruction of G, the distance d between the cycles that are not in the centroid is reconstructible. To find d, we examine all connected cards that are not cycle cards (these are the connected cards whose centroid is not a cycle), and find one where the distance between the two cycles on that card is minimum. This distance is d. Since the centroid of G is the unique block on $G-e$ that has distance less than d to the other two cycles on that card, it must also be the centroid in G'. We have shown that the centroids of G and G' are the same, and therefore G' has two centroidal branches that contain cycles. Next we suppose that G is vertex centroidal. As before, let e be an endline of G, and $G'=G-e+e'$ a reconstruction of G. Let c and c' be the centroids of G and G' respectively. If $c=c'$ then G' has two centroidal branches that contain cycles. Suppose $c \neq c'$. Then G has a centroidal branch B of weight $(|G|-1)/2$, c' is the only vertex of B adjacent to c and the replacing edge e' is incident with a vertex in B other than c. Now if both cycles of G' are contained in a single centroidal branch (at c'), then B must not contain any cycles and consequently G must have two other centroidal branches (at c) that contain cycles. Thus, when the edge cc' is deleted from G', a component of order $(|G|-1)/2$ containing two cycles is formed. But this is impossible since $ED(G)$ has no such edge card. Thus, G' must have two centroidal branches that contain cycles. Finally, if G is edge centroidal then there is an edge card with two components, each with the same number of vertices and each containing a cycle. Clearly, any reconstruction of G will have two centroidal branches with cycles.

Since we know that any reconstruction of G has two centroidal branches with cycles we can reconstruct the centroidal branches of G as follows. We examine those centroidal branches of the cycle cards that contain a cycle, and find one, say B, that has a maximum number of edges. Then any reconstruction of G has a centroidal branch isomorphic to B. Let e be an edge from a cycle of B such that $B-e$ is a centroidal branch of $G-e$. We find all cycle cards that have the minimum number of centroidal branches isomorphic to B possible, and from this set of cards select one with the maximum number of centroidal branches isomorphic to $B-e$ possible. Then any reconstruction of G can be obtained from this card by replacing a branch isomorphic to $B-e$ with one isomorphic to B. Thus, the centroidal branches of G are determined. □

At the end of the proof of the previous proposition we looked for a card which minimized the number of branches of type B and then maximized the number of branches of type $B-e$, and from this card obtained the collection of centroidal branches of G. Similar methods will be used in what follows and it will be convenient to refer to them as *min-max* arguments.

Proposition 4. *If G is vertex centroidal, then G is edge reconstructible.*

Proof. Assume that G is vertex centroidal with centroid c. By Proposition 3 we may assume that G has just one centroidal branch B that contains a cycle. Proposition 3 also implies that any reconstruction of G has only one centroidal branch with a cycle. Let d be the distance from c to the nearest cycle. The graph G has two or more cycles if and only if each cycle card contains a cycle. Thus if G has two or more cycles then

d is the minimum distance from the centroid to a cycle that can be found on a cycle card. Otherwise, G has just one cycle and d is equal to the number of disconnected cards with a component of order $(|G|-1)/2$ or less that contains a cycle. Thus, d is reconstructible. Let e be an endline. If the centroid of $G-e$ is a cyclic block then $d=0$ and c is the root of the largest centroidal branch on that card. Otherwise, $G-e$ is vertex or edge centroidal and c is the vertex in the centroid of $G-e$ that has distance d from a cycle. Thus, we can locate c on any end card. To find B, identify the vertex c on all end cards as described above, and then select an end card where the branch at c that has a cycle has as many edges as possible. This branch is B. Let e be an edge from a cycle of B such that $B-e$ is a centroidal branch of $G-e$. Then, find all cycle cards that have the minimum number of centroidal branches isomorphic to B possible, and from this set of cards select one with the maximum number of centroidal branches isomorphic to $B-e$ possible. Any reconstruction of G can be obtained from this card by replacing a branch isomorphic to $B-e$ with one isomorphic to B. Thus, the centroidal branches of G are determined, and since G is vertex centroidal, this implies that G is reconstructible. □

Proposition 5. *If G is edge centroidal, then G is edge reconstructible.*

Proof. By Proposition 3 we may assume that G has two rooted centroidal branches $A(x)$ and $B(y)$ where A contains a cycle and B does not. We note that all end cards are vertex centroidal. The rooted branch $B(y)$ is a centroidal branch on each cycle card. Also, there must be two edges e_1 and e_2 on a cycle in A where e_1 is nearer to x than e_2, and hence $A(x)-e_1 \not\cong A(x)-e_2$. Thus, the rooted branch $B(y)$ is determined since it is the only type of rooted branch that appears as a centroidal branch on each cycle card. To obtain $A(x)$, find an end card $G-e$ where the distance from the centroid of that card to a cycle is minimum. Then e is an endline of B and the centroid of $G-e$ is x. To obtain $A(x)$ from $G-e$, delete the acyclic branch at x of weight $w(B)$. □

From now on we assume that G is block centroidal with a cyclic centroid C. By Proposition 3 we may assume that G has at most one centroidal branch that contains a cycle.

Proposition 6. *If C is the only cyclic block in G, then the centroidal branches of G are edge reconstructible.*

Proof. Suppose that C is the unique cyclic block of G. Then C is easily identified on all cards $G-e$ where e is a bridge.

Case 1: *All edges not in C are endlines.* In any reconstruction of G all bridges are endlines. If G has just one endline then we are done. Suppose there are exactly two endlines. Then G has one or two centroidal branches depending on whether these endlines are adjacent or not. To determine whether the endlines are adjacent, select from the two end cards one with a cutvertex of maximum degree d. The endlines are adjacent in G if and only if there is a connected card with a cutvertex of degree $d+1$ that is adjacent to two or more endlines. Now suppose that G has three or more endlines. Then G has two or more centroidal branches if and only if some end card has two or more centroidal branches. If there is only one centroidal branch then G is easily seen to be reconstructible. If there are two or more branches then by examining the end cards we can find the weight m of a branch of maximal weight. If we then find an end card of G with a minimum number of centroidal branches of weight m then any reconstruction of G can be obtained from this card by attaching an endline to the root of a branch of weight $m-1$ to form a branch of weight m. Thus the centroidal branches of G are determined.

Case 2: *There is a bridge in G that is not an endline.* The number of branches at C is reconstructible since it is just the maximum number of branches at C that can be found on an end card. Suppose there is just one branch B of C rooted at a vertex v. Then B is a path (with v at the end of the path) if and only if G has just one end card. If G has more than one endline then we can find $B(v)$ as follows. We select a connected card that has a branch at a cut vertex of weight $w(C)$ that has at most one endline. This branch is $C-e$ and the cut vertex is v, and by deleting the branch we are left with B. So $B(v)$ is determined. Now suppose that there is more than one centroidal branch. We look for an end card with a branch B with a maximum number of edges. We then apply a min–max argument to B as in the previous propositions and thus reconstruct the centroidal branches of G. □

Proposition 7. *If G has at least two centroidal branches, then the centroidal branches of G are edge reconstructible.*

Proof. By Propositions 3 and 6 we may assume that exactly one of the centroidal branches contains a cycle. The number of branches at C and their weights can be found by examining a cycle card of G. Suppose that each branch has weight less than $(|G|/2)-1$ so that C is the centroid of each end card of G. If we examine all end cards and cycle cards and find a centroidal branch B with a maximum number of edges, then B is a centroidal branch of G. We can then proceed as in the previous propositions to reconstruct the centroidal branches of G by using a min–max argument.

Thus, we may assume that there is a centroidal branch B with $w(B)=(|G|/2)-1$ and let v be the root of B. Because C has at least three vertices the weight of B is at least 2 greater than the weight of any other centroidal branch unless there are exactly two centroidal branches in which case it is possible that the other branch has weight $w(B)-1$. Note that an end card $G-e$ is either block or vertex centroidal depending on whether e is in B or not.

The branch B is a tree if and only if on every two cycle cards the centroidal branches of weight $w(B)$ found on these cards are isomorphic. If B is a tree then it can be identified as the centroidal branch of maximum weight on any cycle card and thus $B(v)$ is determined. If we select a block centroidal end card $G-e$ then e is an edge of B and $B-e$ is the unique centroidal branch on $G-e$ of maximum weight that has no cycles. By replacing $B-e$ with B on such a card we reconstruct G.

Suppose that B is the unique centroidal branch of G that contains a cycle. We select a cycle card $G-e$ where the centroidal branch of weight $w(B)$ has a minimum number of edges. Then e is an edge of B and the unique centroidal branch of weight $w(B)$ on that card is $B-e$. Let K denote the subgraph of G obtained by deleting all vertices in B except v. Then $K(v)$ can be identified on $G-e$. We reconstruct $B(v)$ as follows. If there exists a vertex centroidal end card with exactly two rooted centroidal branches that are isomorphic, then one of the branches must be B, and $B(v)$ is determined. Otherwise, for every endline x in K, $K(v)-x$ is not isomorphic to $B(v)$. Hence, on any vertex centroidal end card $G-x$ there is a unique centroidal branch isomorphic to an endline deletion of $K(v)$. By deleting this branch we obtain B, and $B(v)$ is determined. Since $B(v)$ and $K(v)$ are both determined, so is G. □

As noted in the introduction, a fundamental result in reconstruction problems is Kelly's Lemma. A corollary to this lemma states that if G and H are graphs where e is an edge of G and H has fewer edges than G, then the number of subgraphs of G that are isomorphic to H and contain e is reconstructible [3,2]. We use this fact in the proof of the following proposition.

Proposition 8. *If G has only one centroidal branch B, then this branch is edge reconstructible.*

Proof. Let v denote the cut vertex at the root of B. By Proposition 6 we can assume that B contains a cycle. If we examine any cycle card we see that $C(v)$ is determined, and if we can show that $B(v)$ is also determined then G is reconstructible. Suppose C is a cycle. Then $ED(G)$ has exactly two edge cards that have a branch at a cutvertex that is a path of length $|C|$. On either card the cutvertex where this branch is rooted is v, and deleting this branch leaves B. Thus $B(v)$ is determined. Next, consider the block-cutvertex tree of G, $BC(G)$. Note that $BC(G)$ is a path if and only if there exists an edge e in C where $BC(G-e)$ is a path. Suppose $BC(G)$ is not a path. From the set of connected cards that are not cycle cards (the cards corresponding to edges deleted from C) we select a card that has a branch at a cut vertex of weight $w(C)$ whose block-cutpoint tree is a path. This branch is $C-e$ and the cut vertex is v, and by deleting the branch we are left with B. So $B(v)$ is determined. Finally, suppose that $BC(G)$ is a path and that C is not a cycle. Then there is a unique endline e in G. Let n be the number of blocks in B. Reconstruct all edge proper subgraphs of G that contain e, have exactly one endline, and have $n+1$ blocks. From these, select one where the number of edges in the n blocks (including the endline) that are nearest to the endline have the maximum number of edges possible. In this subgraph B corresponds to the n blocks nearest the endline, and v corresponds to the cutvertex farthest from the endline. Thus $B(v)$ is determined in all cases. □

We are now ready to state our main result, which follows from the results of the previous section, and from the fact that trees and separable graphs without endlines are reconstructible.

Theorem 1. *If G is a separable graph, then the rooted centroidal branches of G are edge reconstructible.*

We have already seen that vertex centroidal and edge centroidal separable graphs are edge reconstructible. The following corollary to Theorem 1 generalizes this result.

Corollary 1. *If G is a separable graph and the centroid of G is a complete graph, then G is edge reconstructible.*

The next corollary shows that a separable graph is edge reconstructible if the centroidal branches do not have a rather special property.

Corollary 2. *Let G be a separable graph that is not edge reconstructible. Let $B(v)$ be a rooted centroidal branch with at least two edges, and let e be an edge of B such that $G-e$ is either an end card or cycle card. If B' is the component of $B-e$ that contains v, then G has a rooted centroidal branch isomorphic to $B'(v)$.*

Proof. By the previous corollary we may assume that the centroid C of G has order at least four. Let $A(v)$ be a rooted centroidal branch of maximum weight. Let e be an edge of A such that $G-e$ is either a cycle card or end card, and let A' denote the component of $A-e$ that contains v. Since G is not reconstructible there is an edge e' such that $G'=G-e+e'$, G' is a reconstruction of G but $G'\not\approx G$. Since A is a branch of maximum weight and $|C|\geqslant 4$, C is the centroid of both G and G'. By Theorem 1 we know that G and G' have the same set (up to isomorphism) of rooted centroidal branches. Suppose $G-e$ is connected. If e' were incident with vertices of $A-e$, then it would follow that $A(v)-e+e'\approx A(v)$, and hence $G\approx G'$, a contradiction. So $A'(v)=A(v)-e$ is a rooted centroidal branch of G'. Similarly, if $G-e$ is an end card, then e' can not be a bridge between the two components of $A-e$. Thus, e' is not incident with any vertex in A' and $A'(v)$ is a centroidal branch of G'.

We have proved our result for a centroidal branch A of maximum weight. But the only reason we needed to consider a branch of maximum weight was so we could identify the centroid C on $G-e$. We now see that G has one centroidal branch of weight $|A|$ and another of weight at least $|A|-1$, and a centroid of order at least four. Thus, G has no branch of weight $(|G|/2)-1$, and hence C is the centroid of every cycle card and end card. So the same argument we used for A works just as well for an arbitrary centroidal branch B with at least two edges. □

If G is a separable graph that is not reconstructible, then repeated applications of Corollary 2 show that if G has a rooted centroidal branch $B(v)$ with a connected rooted subgraph $B'(v)$, then G has a centroidal branch isomorphic to $B'(v)$. In other words, the centroidal branches of G must be closed under edge deletions. This rather strong condition on the centroidal branches greatly reduces the possibilities for separable counterexamples to the edge reconstruction conjecture.

References

[1] J.A. Bondy, On Ulam's conjecture for separable graphs, Pacific J. Math. 31 (1969) 281–288.

[2] J.A. Bondy, A graph reconstructor's manual, Surveys in combinatorics, 1991 (Guildford, 191), London Math. Soc. Lecture Note Series, Vol. 166, Cambridge Univ. Press, Cambridge, 1991, pp. 221–252.

[3] J.A. Bondy, R.L. Hemminger, Graph reconstruction — A survey, J. Graph Theory 1 (1977) 227–268.

[4] D.L. Greenwell, R.L. Hemminger, Reconstructing graphs, in: G. Chartrand, S.F. Kapoor (Eds.), The many facets of graph theory, Lecture Notes in Mathematics, Vol. 110, Springer, Berlin, 1969, pp. 91–114.

[5] W. Myrvold, The ally-reconstruction number of a tree with five or more vertices is three, J. Graph Theory 14 (1990) 149–166.

[6] Yang Yong Zhi, The reconstruction conjecture is true if all 2-connected graphs are reconstructible, J. Graph Theory 12 (1988) 237–243.